

Hom Gel'fand-Dorfman bialgebras and Hom-Lie conformal algebras ¹

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Abstract. The aim of this paper is to introduce the notions of Hom Gel'fand-Dorfman bialgebra and Hom-Lie conformal algebra. In this paper, we give four constructions of Hom Gel'fand-Dorfman bialgebras. Also, we provide a general construction of Hom-Lie conformal algebras from Hom-Lie algebras. Finally, we prove that a Hom Gel'fand-Dorfman bialgebra is equivalent to a Hom-Lie conformal algebra of degree 2.

Key words: Hom-Novikov algebra, Hom-Lie algebra, Hom Gel'fand-Dorfman bialgebra, Hom-Lie conformal algebra.

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§1. Introduction

The notion of Novikov algebra was originally introduced in connection with the Poisson bracket of hydrodynamic type [3, 6, 7] and Hamiltonian operators in formal variational calculus [11, 27]. The systematic study of Novikov algebras was started by Zel'manov [37] and Filipov [10], whereas the term “Novikov algebra” was first used by Osborn [23]. Novikov algebras constitute a special class of left-symmetric algebras (or under other names like pre-Lie algebras, Vinberg algebras and quasi-associative algebras), arising from the study of affine manifolds, affine structures and convex homogeneous cones [2, 16, 25]. Left-symmetric algebras are closely related with many important fields in mathematics and mathematical physics, such as infinite-dimensional Lie algebras [4], classical and quantum Yang-Baxter equation [9, 12], and quantum field theory [5]. The superanalogue of Novikov algebras was introduced and studied by Xu [29, 30].

Hom-Lie algebras were initially introduced by Hartwig, Larsson and Silvestrov [13] motivated by constructing examples of deformed Lie algebras coming from twisted discretizations of vector fields. This kind of algebraic structure contains Lie algebras as a particular case. Hom-Lie algebras were intensively discussed in [31, 32, 33, 34]. Hom-associative algebras were introduced in [21], where it was shown that the commutator bracket of a Hom-associative algebra gives rise to a Hom-Lie algebra. The graded case was mentioned in [17, 18], while a detailed study was given in [1, 36].

Gelfand and Dorfman [11] introduced the notion of Gel'fand-Dorfman bialgebra in the study of certain Hamiltonian pairs, which play important roles in complete integrability of nonlinear evolution partially differential equations. The super Gel'fand-Dorfman bialgebra was introduced and studied in [26], where it was proved that a super Gel'fand-Dorfman bialgebra is equivalent to a quadratic conformal superalgebra. An analogous result for Gel'fand-Dorfman bialgebras was essentially stated in [11].

The notion of Lie conformal (super)algebra, introduced by Kac [14], encodes an axiomatic description of the operator product expansions of chiral fields in conformal field theory. Closely related to vertex algebras [19, 20], Lie conformal algebras have important applications in other

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areas of algebras and integrable systems [38, 39]. In particular, they provide powerful tools in the study of infinite-dimensional Lie (super)algebras satisfying the locality property [14, 24]. The main examples are those based on the punctured complex plane, such as the Virasoro algebra and the loop Lie algebras [8].

In this paper, we introduce and study Hom Gel'fand-Dorfman bialgebras and Hom-Lie conformal algebras, which are natural generalizations of Gel'fand-Dorfman bialgebras and Lie conformal algebras, respectively.

The paper is organized as follows: In Sec. 2, we review the notions of Novikov algebra, Hom-Lie algebra, Hom-Novikov algebra, Gel'fand-Dorfman algebra and Lie conformal algebra. In Sec. 3, we introduce the notion of Hom Gel'fand-Dorfman bialgebra and give four basic constructions of Hom Gel'fand-Dorfman bialgebras. In Sec. 4, we introduce the notion of Hom-Lie conformal algebra and provide a general construction of Hom-Lie conformal algebras. In Sec. 5, we prove that a Hom Gel'fand-Dorfman bialgebra is equivalent to a Hom-Lie conformal algebra of degree 2.

§2. Preliminaries

In this section, we recall the notions of Novikov algebra, Hom-Novikov algebra, Hom-Lie algebra, Gel'fand-Dorfman algebra and Lie conformal algebra. Throughout this paper, all vector spaces and tensor products are over the complex field \mathbb{C} . In addition to the standard notation \mathbb{Z} , we use \mathbb{Z}^+ to denote the set of nonnegative integers.

Definition 2.1 A Novikov algebra is a vector space \mathcal{A} equipped with an operation \circ such that for $x, y, z \in \mathcal{A}$:

$$(x \circ y) \circ z = (x \circ z) \circ y, \quad (x, y, z) = (y, x, z), \quad (2.1)$$

where the associator $(x, y, z) = (x \circ y) \circ z - x \circ (y \circ z)$.

The first identity in (2.1) says that the right multiplication operators on \mathcal{A} commute with one another and the second identity states that the associator is symmetric in the first two variables.

Yau introduced a twisted generalization of Novikov algebras, named Hom-Novikov algebras [35].

Definition 2.2 A Hom-Novikov algebra is a vector space \mathcal{A} equipped with a bilinear operation \circ and a linear endomorphism α , such that the following two relations hold for $x, y, z \in \mathcal{A}$:

$$(x \circ y) \circ \alpha(z) - \alpha(x) \circ (y \circ z) = (y \circ x) \circ \alpha(z) - \alpha(y) \circ (x \circ z), \quad (2.2)$$

$$(x \circ y) \circ \alpha(z) = (x \circ z) \circ \alpha(y). \quad (2.3)$$

Obviously, Novikov algebras are examples of Hom-Novikov algebras where $\alpha = \text{id}$. The study of Hom-type algebras started with the notion of Hom-Lie algebra, which was originally introduced by Hartwig, Larsson and Silvestrov [13] motivated by constructing deformations of the Witt and Virasoro algebras based on σ -derivations.

Definition 2.3 A Hom-Lie algebra is a vector space L with a bilinear map $[\cdot, \cdot] : L \times L \longrightarrow L$ and a linear map $\alpha : L \longrightarrow L$, such that the following relations hold for $x, y, z \in L$:

$$[x, y] = -[y, x], \quad (\text{skew-symmetry}) \quad (2.4)$$

$$[[x, y], \alpha(z)] + [[y, z], \alpha(x)] + [[z, x], \alpha(y)] = 0. \quad (\text{Hom-Jacobi identity}) \quad (2.5)$$

The notion of Gel'fand-Dorfman bialgebra was initially introduced in the study of Hamilton operators [11].

Definition 2.4 A Gel'fand-Dorfman bialgebra is a vector space \mathcal{A} , equipped with two operations $[\cdot, \cdot]$ and \circ such that $(\mathcal{A}, [\cdot, \cdot])$ forms a Lie algebra, (\mathcal{A}, \circ) forms a Novikov algebra and the compatibility condition holds for $x, y, z \in \mathcal{A}$:

$$[x \circ y, z] - [x \circ z, y] + [x, y] \circ z - [x, z] \circ y - x \circ [y, z] = 0. \quad (2.6)$$

Such a bialgebraic structure corresponds to the following Poisson bracket of dynamic type

$$[u(x), v(y)] = [u, v](x)\delta(x, y) + \partial_x(u \circ v)(x)\delta(x, y) + (u \circ v + v \circ u)(x)\partial_x\delta(x, y). \quad (2.7)$$

It was pointed out in [11] that if we define

$$[x, y]^- = x \circ y - y \circ x, \text{ for } x, y \in \mathcal{A}, \quad (2.8)$$

for a Novikov algebra (\mathcal{A}, \circ) , then $(\mathcal{A}, [\cdot, \cdot]^-, \circ)$ forms a Gel'fand-Dorfman bialgebra.

The follow definition is due to Kac [14].

Definition 2.5 A Lie conformal algebra is a $\mathbb{C}[\partial]$ -module \mathcal{R} with a λ -bracket $[\cdot, \cdot]_\lambda$ which defines a \mathbb{C} -bilinear map from $\mathcal{R} \otimes \mathcal{R}$ to $\mathcal{R}[\lambda]$, where $\mathcal{R}[\lambda] = \mathbb{C}[\lambda] \otimes \mathcal{R}$ is the space of polynomials of λ with coefficients in \mathcal{R} , satisfying the following axioms for all $a, b, c \in \mathcal{R}$:

$$[\partial a_\lambda b] = -\lambda[a_\lambda b], \quad (\text{conformal sesquilinearity}) \quad (2.9)$$

$$[a_\lambda b] = -[b_{-\lambda-\partial} a], \quad (\text{skew-symmetry}) \quad (2.10)$$

$$[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]]. \quad (\text{Jacobi identity}) \quad (2.11)$$

Let m be a positive integer. A Lie conformal algebra \mathcal{R} is called a *Lie conformal algebra of degree m* if for any $a, b \in \mathcal{R}$,

$$[a_\lambda b] = \sum_{j \geq 0, i+j < m} \partial^i w_{i,j} \lambda^j, \text{ for some } w_{i,j} \in \mathcal{R}. \quad (2.12)$$

It was stated in [11] that a Gel'fand-Dorfman bialgebra is equivalent to a Lie conformal algebra of degree 2. This statement was proved by Xu [26] in the super case.

§3. Hom Gel'fand-Dorfman bialgebras

In this section, we introduce the notion of Hom Gel'fand-Dorfman bialgebra. Also, we present four constructions of Hom Gel'fand-Dorfman bialgebras from Hom-Novikov algebras, Gel'fand-Dorfman bialgebras, commutative Hom-associative algebras and Hom-Poisson algebras along with some suitable algebra endomorphisms or derivations.

Definition 3.1 A Hom Gel'fand-Dorfman bialgebra is a vector space \mathcal{A} equipped with a linear endomorphism α and two operations $[\cdot, \cdot]$ and \circ , such that $(\mathcal{A}, [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra, $(\mathcal{A}, \circ, \alpha)$ is a Hom-Novikov algebra and the following compatibility condition holds for $x, y, z \in \mathcal{A}$:

$$[x \circ y, \alpha(z)] - [x \circ z, \alpha(y)] + [x, y] \circ \alpha(z) - [x, z] \circ \alpha(y) - \alpha(x) \circ [y, z] = 0. \quad (3.1)$$

Clearly, we recover the Gel'fand-Dorfman bialgebras when $\alpha = \text{id}$. In the sequel, we shall simply call a Hom Gel'fand-Dorfman bialgebra a Hom-GD bialgebra.

Our first construction of Hom-GD bialgebras is from Hom-Novikov algebras, analogous to the fundamental construction of Gel'fand-Dorfman bialgebras from Novikov algebras via the commutator bracket [11].

Theorem 3.2 *Let $(\mathcal{A}, \circ, \alpha)$ be a Hom-Novikov algebra. Define the commutator*

$$[x, y]^- = x \circ y - y \circ x, \quad \text{for } x, y \in \mathcal{A}. \quad (3.2)$$

Then $(\mathcal{A}, [\cdot, \cdot]^-, \circ, \alpha)$ forms a Hom-GD algebra.

Proof. Clearly, $[\cdot, \cdot]^-$ is skew-symmetric. For any $x, y, z \in \mathcal{A}$, we have

$$\begin{aligned} & [x, y]^- \circ \alpha(z) + [y, z]^- \circ \alpha(x) + [z, x]^- \circ \alpha(y) \\ &= ((x \circ y) \circ \alpha(z) - (x \circ z) \circ \alpha(y)) + ((y \circ z) \circ \alpha(x) - (y \circ x) \circ \alpha(z)) \\ & \quad + ((z \circ x) \circ \alpha(y) - (z \circ y) \circ \alpha(x)) \\ &= 0, \end{aligned} \quad (3.3)$$

by (2.3) and (3.2). Similarly, we get

$$\alpha(x) \circ [y, z] + \alpha(y) \circ [z, x] + \alpha(z) \circ [x, y] = 0, \quad (3.4)$$

which together with (3.3) implies (2.5). Thus $(\mathcal{A}, [\cdot, \cdot]^-, \alpha)$ is a Hom-Lie algebra. By (2.2) and (3.2), we obtain

$$\begin{aligned} & [x \circ y, \alpha(z)]^- - [x \circ z, \alpha(y)]^- + [x, y]^- \circ \alpha(z) - [x, z]^- \circ \alpha(y) - \alpha(x) \circ [y, z]^- \\ &= (x \circ y) \circ \alpha(z) - \alpha(z) \circ (x \circ y) - (x \circ z) \circ \alpha(y) + \alpha(y) \circ (x \circ z) + (x \circ y) \circ \alpha(z) \\ & \quad - (y \circ x) \circ \alpha(z) - (x \circ z) \circ \alpha(y) + (z \circ x) \circ \alpha(y) - \alpha(x) \circ (y \circ z) + \alpha(x) \circ (z \circ y) \\ &= ((x \circ y) \circ \alpha(z) + \alpha(y) \circ (x \circ z) - (y \circ x) \circ \alpha(z) - \alpha(x) \circ (y \circ z)) \\ & \quad + ((z \circ x) \circ \alpha(y) + \alpha(x) \circ (z \circ y) - \alpha(z) \circ (x \circ y) - (x \circ z) \circ \alpha(y)) \\ &= 0. \end{aligned}$$

This proves (3.1) and the theorem. □

The following result, due to Yau [35], gives a construction of Hom-Novikov algebras from Novikov algebras with an algebra endomorphism.

Proposition 3.3 *Let (\mathcal{A}, \circ) be a Novikov algebra with an algebra endomorphism α . Then $(\mathcal{A}, \circ_\alpha, \alpha)$ forms a Hom-Novikov algebra, where*

$$x \circ_\alpha y = \alpha(x) \circ \alpha(y), \quad \text{for } x, y \in \mathcal{A}. \quad (3.5)$$

As a consequence of Theorem 3.2 and Proposition 3.3, we obtain

Corollary 3.4 *Let (\mathcal{A}, \circ) be a Novikov algebra and α an algebra endomorphism of (\mathcal{A}, \circ) . Then $(\mathcal{A}, [\cdot, \cdot]_{\alpha}^{-}, \circ_{\alpha}, \alpha)$ is a Hom-GD bialgebra, with \circ_{α} defined by (3.5) and*

$$[x, y]_{\alpha}^{-} = \alpha(x \circ y) - \alpha(y \circ x), \text{ for } x, y \in \mathcal{A}. \quad (3.6)$$

The following theorem, which is an analogue of Proposition 3.3, shows our second construction of Hom-GD bialgebras from a Gel'fand-Dorfman algebra with an algebra endomorphism.

Theorem 3.5 *Let $(\mathcal{A}, [\cdot, \cdot], \circ)$ be a Gel'fand-Dorfman algebra with an algebra endomorphism α . Then $(\mathcal{A}, [\cdot, \cdot]_{\alpha}, \circ_{\alpha}, \alpha)$ forms a Hom-GD bialgebra, where \circ_{α} is defined by (3.5) and*

$$[x, y]_{\alpha} = [\alpha(x), \alpha(y)], \text{ for } x, y \in \mathcal{A}. \quad (3.7)$$

Proof. Since $(\mathcal{A}, [\cdot, \cdot]_{\alpha}, \alpha)$ is a Hom-Lie algebra and $(\mathcal{A}, \circ_{\alpha}, \alpha)$ is a Hom-Novikov algebra, it remains to check the compatibility condition (3.1). As α is an algebra homomorphism of $(\mathcal{A}, [\cdot, \cdot], \circ, \alpha)$ and by (2.6), we have

$$\begin{aligned} & [x \circ_{\alpha} y, \alpha(z)]_{\alpha} - [x \circ_{\alpha} z, \alpha(y)]_{\alpha} + [x, y]_{\alpha} \circ_{\alpha} \alpha(z) - [x, z]_{\alpha} \circ_{\alpha} \alpha(y) - \alpha(x) \circ_{\alpha} [y, z]_{\alpha} \\ &= \alpha^2([x \circ y, z] - [x \circ z, y] + [x, y] \circ z - [x, z] \circ y - x \circ [y, z]) = 0, \end{aligned}$$

which concludes the proof. \square

Example 3.6 Suppose that Δ is an additive subgroup of \mathbb{C} and denote $\Gamma = \{0, 1, 2, \dots\}$. Let $\mathcal{A}_{\Delta, \Gamma}$ be the vector space with a basis $\{x_{a,j} | (a, j) \in \Delta \times \Gamma\}$. For any given $\xi \in \mathbb{C}$, define an algebra operation on $\mathcal{A}_{\Delta, \Gamma}$ by

$$x_{a,i} \circ x_{b,j} = (b + \xi)x_{a+b, i+j} + jx_{a+b, i+j-1}, \text{ for } a, b \in \Delta, i, j \in \Gamma.$$

It was proved in [28] that $(\mathcal{A}_{\Delta, \Gamma}, \circ)$ forms a simple Novikov algebra. According to Gel'fand and Dorfman's statement in [11], there is a Gel'fand-Dorfman bialgebra $(\mathcal{A}_{\Delta, \Gamma}, [\cdot, \cdot]^{-}, \circ)$ with

$$\begin{aligned} [x_{a,i}, x_{b,j}]^{-} &= x_{a,i} \circ x_{b,j} - x_{b,j} \circ x_{a,i} \\ &= (b - a)x_{a+b, i+j} + (j - i)x_{a+b, i+j-1}, \end{aligned}$$

for $a, b \in \Delta, i, j \in \Gamma$. Define a linear endomorphism α of $\mathcal{A}_{\Delta, \Gamma}$ by

$$\alpha(x_{a,j}) = e^a x_{a,j}, \text{ for } a \in \Delta. \quad (3.8)$$

It is easy to check that α is an algebra homomorphism of $(\mathcal{A}_{\Delta, \Gamma}, [\cdot, \cdot]^{-}, \circ)$. By Theorem 3.5, we get a Hom-GD bialgebra $(\mathcal{A}_{\Delta, \Gamma}, [\cdot, \cdot]_{\alpha}^{-}, \circ_{\alpha}, \alpha)$.

Let (\mathcal{A}, \cdot) be a commutative associative algebra with a derivation D . It was proved that the new operation

$$x \circ y = x \cdot D(y) + \lambda x \cdot y, \text{ for } x, y \in \mathcal{A}, \quad (3.9)$$

equips \mathcal{A} with a structure of Novikov algebra for $\lambda = 0$ by Gel'fand and Dorfman [11], for $\lambda \in \mathbb{F}$ by Filipov [10], and for a fixed element $\lambda \in \mathcal{A}$ by Xu [27]. Yau [35] gave an analogous construction of Hom-Novikov algebras (in $\lambda = 0$ setting) from Hom-associative algebras introduced in [21].

Definition 3.7 A Hom-associative algebra is a vector space V with a bilinear map $\mu : V \times V \longrightarrow V$ and a linear map $\alpha : V \longrightarrow V$, such that

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)), \text{ for } x, y, z \in \mathcal{A}. \quad (3.10)$$

A Hom-associative algebra $(\mathcal{A}, \mu, \alpha)$ is called *commutative* if $\mu(x, y) = \mu(y, x)$ for $x, y \in \mathcal{A}$. A *derivation* on a Hom-associative algebra is defined in the usual way.

The following result presents our third construction of Hom-GD bialgebras from a commutative Hom-associative algebra with a derivation.

Theorem 3.8 *Let $(\mathcal{A}, \cdot, \alpha)$ be a commutative Hom-associative algebra with a derivation D such that $\alpha D = D\alpha$. Then $(\mathcal{A}, [\cdot, \cdot]^-, \circ, \alpha)$ forms a Hom-GD algebra, where \circ is defined by (3.9) and*

$$[x, y]^- = x \cdot D(y) - y \cdot D(x), \text{ for } x, y \in \mathcal{A}. \quad (3.11)$$

Proof. By (3.9), (3.10) and the fact that $\alpha D = D\alpha$, we have

$$\begin{aligned} (x \circ y) \circ \alpha(z) &= (x \cdot D(y) + \lambda x \cdot y) \circ \alpha(z) \\ &= (x \cdot D(y) + \lambda x \cdot y) \cdot D(\alpha(z)) + \lambda(x \cdot D(y) + \lambda x \cdot y) \cdot \alpha(z) \\ &= \alpha(x) \cdot (D(y) \cdot D(z)) + \lambda \alpha(x) \cdot (y \cdot D(z) + D(y) \cdot z) + \lambda^2 \alpha(x) \cdot (y \cdot z). \end{aligned} \quad (3.12)$$

Similarly, we have

$$(x \circ z) \circ \alpha(y) = \alpha(x) \cdot (D(z) \cdot D(y)) + \lambda \alpha(x) \cdot (z \cdot D(y) + D(z) \cdot y) + \lambda^2 \alpha(x) \cdot (z \cdot y). \quad (3.13)$$

Combining (3.12) with (3.13) and using commutativity of $(\mathcal{A}, \cdot, \alpha)$, we get

$$(x \circ y) \circ \alpha(z) = (x \circ z) \circ \alpha(y),$$

which proves (2.3). By commutativity and Hom-associativity of $(\mathcal{A}, \cdot, \alpha)$, one can obtain

$$(x \circ y) \circ \alpha(z) - \alpha(x) \circ (y \circ z) = -(x \cdot y) \cdot \alpha(D^2(z)) - \lambda(x \cdot y) \cdot \alpha(D(z)), \quad (3.14)$$

which implies (2.2), because the right-hand side of (3.14) is symmetric in x and y . So $(\mathcal{A}, \circ, \alpha)$ is a Hom-Novikov algebra. By (3.11) and commutativity of (\mathcal{A}, \cdot) , we have $[x, y]^- = x \circ y - y \circ x$. It follows from Theorem 3.2 that $(\mathcal{A}, [\cdot, \cdot]^-, \circ, \alpha)$ is a Hom-GD algebra. \square

From the proof of the above theorem, we have the following result, generalizing Yau's construction of Hom-Novikov algebras (see [35, Theorem 1.2]).

Corollary 3.9 *Let $(\mathcal{A}, \cdot, \alpha)$ be a commutative Hom-associative algebra with a derivation D such that $\alpha D = D\alpha$. For any fixed number λ , define*

$$x \circ y = x \cdot D(y) + \lambda x \cdot y, \text{ for } x, y \in \mathcal{A}. \quad (3.15)$$

Then $(\mathcal{A}, \circ, \alpha)$ is a Hom-Novikov algebra.

Moreover, we have

Corollary 3.10 *Let (\mathcal{A}, \cdot) be a commutative associative algebra with an algebra endomorphism α and a derivation D , such that $D\alpha = \alpha D$. For any fixed number λ , define*

$$x \circ y = \alpha(x \cdot D(y)) + \lambda \alpha(x \cdot y), \text{ for } x, y \in \mathcal{A}. \quad (3.16)$$

Then $(\mathcal{A}, [\cdot, \cdot]^{-}, \circ, \alpha)$ forms a Hom-GD bialgebra, where

$$[x, y]^{-} = \alpha(x \cdot D(y)) - \alpha(y \cdot D(x)), \text{ for } x, y \in \mathcal{A}. \quad (3.17)$$

Proof. For clarity, write $x \cdot_{\alpha} y = \alpha(x \cdot y)$ for $x, y \in \mathcal{A}$. Because α is an algebra endomorphism of (\mathcal{A}, \cdot) , we have $(\mathcal{A}, \cdot_{\alpha}, \alpha)$ is a commutative Hom-associative algebra. Since $\alpha D = D\alpha$, D is also a derivation of $(\mathcal{A}, \cdot_{\alpha})$. We can rewrite (3.16) as

$$x \circ y = x \cdot_{\alpha} D(y) + \lambda x \cdot_{\alpha} y \text{ for } x, y \in \mathcal{A}.$$

According to Theorem 3.8, $(\mathcal{A}, [\cdot, \cdot]^{-}, \circ, \alpha)$ forms a Hom-GD bialgebra. □

The following three examples are applications of Corollary 3.10.

Example 3.11 Consider the polynomial algebra $\mathbb{C}[x]$. Let $D = \frac{d}{dx}$ be the differential operator and α an endomorphism of $\mathbb{C}[x]$ determined by

$$\alpha(x^n) = (x + c)^n, \text{ with } c \in \mathbb{C}.$$

One has $\alpha D = D\alpha$ (cf. [35]). By Corollary 3.10, we have a Hom-GD bialgebra $(\mathbb{C}[x], [\cdot, \cdot]^{-}, \circ, \alpha)$ with

$$\begin{aligned} f(x) \circ g(x) &= f(x + c)D(g(x + c)) + \lambda f(x + c)g(x + c), \\ [f(x), g(x)]^{-} &= f(x + c)D(g(x + c)) - g(x + c)D(f(x + c)), \end{aligned}$$

for all $f(x), g(x) \in \mathbb{C}[x]$, where λ is a fixed number.

Example 3.12 Let A be a commutative associative algebra with a nilpotent derivation D , namely, there exists a positive integer n such that $D^n = 0$. The formal exponential map

$$\alpha = e^D = \text{id} + D + \frac{1}{2!}D^2 + \cdots + \frac{1}{(n-1)!}D^{n-1}$$

is an algebra automorphism of A , and $\alpha D = D\alpha$ (cf. [35]). For a fixed number λ , the operations

$$x \circ y = \alpha(x \cdot D(y)) + \lambda \alpha(x \cdot y), \quad [x, y]^{-} = \alpha(x \cdot D(y)) - \alpha(y \cdot D(x)), \text{ for } x, y \in A,$$

give a Hom-GD bialgebra $(A, [\cdot, \cdot]^{-}, \circ, \alpha)$ by Corollary 3.10.

Example 3.13 Let Δ be a nonzero abelian subgroup of \mathbb{C} . Suppose that f is a nontrivial homomorphism of Δ into the additive group of \mathbb{C} . Consider the vector space \mathcal{A} with a basis $\{x_a | a \in \Delta\}$. Define an operation on \mathcal{A} by

$$x_a \cdot x_b = x_{a+b}, \quad \text{for all } a, b \in \Delta.$$

Then (\mathcal{A}, \cdot) is a commutative associative algebra. Define a linear map $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ by

$$\alpha(x_a) = e^a x_a, \quad \text{for all } a \in \Delta.$$

Then α is an algebra homomorphism of (\mathcal{A}, \cdot) . Define another linear map $\partial : \mathcal{A} \rightarrow \mathcal{A}$ by

$$\partial(x_a) = f(a)x_a, \quad \text{for all } a \in \Delta.$$

It is easy to check that ∂ is a derivation of (\mathcal{A}, \cdot) and $\alpha\partial = \partial\alpha$. By Corollary 3.10, we obtain a Hom-GD bialgebra $(\mathcal{A}, [\cdot, \cdot]^-, \circ, \alpha)$ with

$$x_a \circ x_b = (f(b) + \lambda)e^{a+b}x_{a+b}, \quad [x_a, x_b]^- = f(a+b)e^{a+b}x_{a+b},$$

for all $a, b \in \Delta$, where λ is a fixed number.

Our fourth construction of Hom-GD bialgebras is related to the notion of Hom-Poisson algebra, which was introduced in the study of deformation theory of Hom-Lie algebras [22].

Definition 3.14 A Hom-Poisson algebra is a vector space \mathcal{A} equipped with two operations \cdot and $[\cdot, \cdot]$, and a linear endomorphism α , such that $(\mathcal{A}, \cdot, \alpha)$ is a commutative Hom-associative algebra, $(\mathcal{A}, [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra, and the following relation holds for $x, y, z \in \mathcal{A}$:

$$[\alpha(x), y \cdot z] = \alpha(y) \cdot [x, z] + \alpha(z) \cdot [x, y]. \quad (3.18)$$

Notice that (3.18) can be equivalently reformulated as

$$[x \cdot y, \alpha(z)] = [x, z] \cdot \alpha(y) + \alpha(x) \cdot [y, z]. \quad (3.19)$$

By setting $\alpha = \text{id}$ in above definition, we recover Lie-Poisson algebras, which appear naturally in Hamiltonian mechanics, and are central in the study of quantum groups.

The following result can be considered as a Hom-version of [26, Theorem 3.2].

Theorem 3.15 Let $(\mathcal{A}, \cdot, [\cdot, \cdot], \alpha)$ be a Hom-Poisson algebra and D a derivation of (\mathcal{A}, \cdot) , such that $D\alpha = \alpha D$ and

$$D([x, y]) = [D(x), y] + [x, D(y)] + \lambda[x, y], \quad \text{for } x, y \in \mathcal{A}, \quad (3.20)$$

where λ is a fixed number. Define a new operation \circ by

$$x \circ y = x \cdot D(y) + \lambda x \cdot y, \quad \text{for } x, y \in \mathcal{A}. \quad (3.21)$$

Then $(\mathcal{A}, [\cdot, \cdot], \circ, \alpha)$ forms a Hom-GD bialgebra.

Proof. By Theorem 3.8, $(\mathcal{A}, \circ, \alpha)$ is a Hom-Novikov algebra. By (3.18)–(3.21) and the fact that D is a derivation of (\mathcal{A}, \cdot) commuting with α , we get

$$\begin{aligned}
& [x \circ y, \alpha(z)] - [x \circ z, \alpha(y)] + [x, y] \circ \alpha(z) - [x, z] \circ \alpha(y) - \alpha(x) \circ [y, z] \\
&= [x \cdot D(y) + \lambda x \cdot y, \alpha(z)] - [x \cdot D(z) + \lambda x \cdot z, \alpha(y)] + [x, y] \cdot \alpha(D(z)) + \lambda [x, y] \cdot \alpha(z) \\
&\quad - [x, z] \cdot \alpha(D(y)) - \lambda [x, z] \cdot \alpha(y) - \alpha(x) \cdot D([y, z]) - \lambda \alpha(x) \cdot [y, z] \\
&= [x \cdot D(y), \alpha(z)] - [x \cdot D(z) + \lambda x \cdot z, \alpha(y)] + [x, y] \cdot \alpha(D(z)) + \lambda [x, y] \cdot \alpha(z) \\
&\quad - [x, z] \cdot \alpha(D(y)) - \alpha(x) \cdot ([D(y), z] + [y, D(z)] + \lambda [y, z]) \\
&= ([x \cdot D(y), \alpha(z)] - [x, z] \cdot \alpha(D(y)) - \alpha(x) \cdot [D(y), z]) \\
&\quad - ([x \cdot D(z), \alpha(y)] - [x, y] \cdot \alpha(D(z)) - \alpha(x) \cdot [D(z), y]) = 0,
\end{aligned}$$

which proves (3.1) and the theorem. \square

As it was explained in [26], the above construction is related to Lie superalgebras of Hamiltonian type and Contact type [15].

§4. Hom-Lie conformal algebras

In this section, we introduce the notion of Hom-Lie conformal algebra and give a general construction of Hom-Lie conformal algebras from formal distribution Hom-Lie algebras, analogous to the construction of Lie conformal algebras from formal distribution Lie algebras.

Definition 4.1 A Hom-Lie conformal algebra is a $\mathbb{C}[\partial]$ -module \mathcal{R} equipped with a linear endomorphism α such that $\alpha\partial = \partial\alpha$, and a λ -bracket $[\cdot_\lambda \cdot]$ which defines a \mathbb{C} -bilinear map from $\mathcal{R} \otimes \mathcal{R}$ to $\mathcal{R}[\lambda] = \mathbb{C}[\lambda] \otimes \mathcal{R}$ such that the following axioms hold for $a, b, c \in \mathcal{R}$:

$$[\partial a_\lambda b] = -\lambda[a_\lambda b], \quad (\text{conformal sesquilinearity}) \quad (4.1)$$

$$[a_\lambda b] = -[b_{-\lambda-\partial} a], \quad (\text{skew-symmetry}) \quad (4.2)$$

$$[\alpha(a)_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} \alpha(c)] + [\alpha(b)_\mu [a_\lambda c]]. \quad (\text{Hom-Jacobi identity}) \quad (4.3)$$

Remark 4.2 We recover Lie conformal algebras when $\alpha = \text{id}$. In addition, we can associate to any Lie conformal algebra a Hom-Lie conformal algebra structure by taking $\alpha = 0$. Such a Hom-Lie conformal algebra is called *trivial*.

Remark 4.3 It follows from (4.1) and (4.2) that

$$[a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b], \quad \text{for } a, b \in \mathcal{R}. \quad (4.4)$$

Thus ∂ acts as a derivation on the λ -bracket.

A Hom-Lie conformal algebra \mathcal{R} is called *finite* if \mathcal{R} is a finitely generated $\mathbb{C}[\partial]$ -module. The *rank* of \mathcal{R} is its rank as a $\mathbb{C}[\partial]$ -module. Let m be a positive integer. \mathcal{R} is called a *Hom-Lie conformal algebra of degree m* if for any $a, b \in \mathcal{R}$, there exist $w_{i,j} \in \mathcal{R}$ such that

$$[a_\lambda b] = \sum_{j \geq 0, i+j < m} \partial^i w_{i,j} \lambda^j. \quad (4.5)$$

In particular, for $m = 2$, there exist $c_{0,0}, c_{1,0}, c_{0,1} \in \mathcal{R}$ such that

$$[a_\lambda b] = c_{0,0} + \partial c_{1,0} + \lambda c_{0,1}. \quad (4.6)$$

Let V be any vector space. Following [14], a V -valued formal distribution is a formal series

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, \quad (4.7)$$

where $a_{(n)} = \text{Res}_z z^n a(z) \in V$ is called the *Fourier coefficient* of $a(z)$. The notion of *residue* is taken formally in analogy with Complex Analysis. The vector space of these formal distributions in one variable is denoted by $V[[z, z^{-1}]]$. Similarly, a formal distribution $a(z, w)$ in two variables is defined as a series $\sum_{m,n \in \mathbb{Z}} a_{m,n} z^{-m-1} w^{-n-1}$ and the space of these series is denoted by $V[[z, z^{-1}, w, w^{-1}]]$. The *delta distribution* is the \mathbb{C} -valued formal distribution

$$\delta(z, w) = \sum_{n \in \mathbb{Z}} z^{-n-1} w^n. \quad (4.8)$$

It enjoys the following properties:

$$\partial_z \delta(z, w) = -\partial_w \delta(z, w), \quad (z - w)^m \partial_w^n \delta(z, w) = 0 \quad \text{for } m > n. \quad (4.9)$$

It is known that operator product expansions (OPE's) are widely used in conformal field theory. The fundamental notion beneath it is the locality of formal distributions in two variables.

Definition 4.4 Let $a(z, w) \in V[[z, z^{-1}, w, w^{-1}]]$. The formal distribution $a(z, w)$ is called local if there exists a positive integer N such that $(z - w)^N a(z, w) = 0$.

The following decomposition theorem is due to Kac [14].

Theorem 4.5 Let $a(z, w) \in V[[z, z^{-1}, w, w^{-1}]]$ be a local formal distribution. Then $a(z, w)$ can be written as a finite sum of $\delta(z, w)$ and its derivatives:

$$a(z, w) = \sum_{j \in \mathbb{Z}^+} c^j(w) \frac{\partial_w^j \delta(z, w)}{j!}, \quad (4.10)$$

where $c^j(w) = \text{Res}_z (z - w)^j a(z, w) \in V[[w, w^{-1}]]$. In addition, the converse is true.

Denote

$$e^{\lambda(z-w)} = \sum_{j \in \mathbb{Z}^+} \frac{\lambda^j}{j!} (z - w)^j \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]][[\lambda]]. \quad (4.11)$$

The formal Fourier transform of $a(z, w) \in V[[z, z^{-1}, w, w^{-1}]]$ is defined by

$$F_{z,w}^\lambda a(z, w) = \text{Res}_z e^{\lambda(z-w)} a(z, w). \quad (4.12)$$

Notice that $F_{z,w}^\lambda$ is a linear map from $V[[z, z^{-1}, w, w^{-1}]]$ to $V[[w, w^{-1}]][[\lambda]]$. Indeed, we have

$$F_{z,w}^\lambda a(z, w) = \sum_{j \in \mathbb{Z}^+} \frac{\lambda^j}{j!} c^j(w), \quad \text{where } c^j(w) = \text{Res}_z (z - w)^j a(z, w). \quad (4.13)$$

If $a(z, w)$ is local, then $F_{z,w}^\lambda a(z, w) \in V[[w, w^{-1}]][[\lambda]]$, and

$$F_{z,w}^\lambda \partial_z a(z, w) = -\lambda F_{z,w}^\lambda a(z, w) = [\partial_w, F_{z,w}^\lambda], \quad F_{z,w}^\lambda a(w, z) = F_{z,w}^\mu a(z, w)|_{\mu=-\lambda-\partial_w}. \quad (4.14)$$

Let $(\mathcal{L}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra. We can extend the Hom-Lie bracket to the commutator between two \mathcal{L} -valued formal distributions $a(z) = \sum_{m \in \mathbb{Z}} a_m z^{-m-1}$ and $b(w) = \sum_{n \in \mathbb{Z}} b_n w^{-n-1}$ by

$$[a(z), b(w)] = \sum_{m, n \in \mathbb{Z}} [a_{(m)}, b_{(n)}] z^{-m-1} w^{-n-1} \in \mathcal{L}[[z, z^{-1}, w, w^{-1}]]. \quad (4.15)$$

If $[a(z), b(w)]$ is local, we say $a(z)$ and $b(w)$ are *mutually local*, or $(a(z), b(w))$ is a *local pair*. Taking the formal Fourier transform $F_{z,w}^\lambda$ defined by (4.12) on both parts of (4.15), we can define a \mathbb{C} -bilinear map $[\cdot]_\lambda$, called a λ -bracket, from $\mathcal{L}[[w, w^{-1}]] \otimes \mathcal{L}[[w, w^{-1}]]$ to $\mathcal{L}[[w, w^{-1}]][[\lambda]]$ by

$$[a(w)]_\lambda b(w) = F_{z,w}^\lambda [a(z), b(w)]. \quad (4.16)$$

In the sequel, the λ -bracket between $a(w)$ and $b(w)$ will be simply denoted by $[a_\lambda b]$.

Let $(a(z), b(w))$ be a local pair of \mathcal{L} -valued formal distributions. By Theorem 4.5, we have

$$[a(z), b(w)] = \sum_{j \in \mathbb{Z}^+} a(w)_{(j)} b(w) \frac{\partial_w^j \delta(z, w)}{j!}, \quad (4.17)$$

where

$$a(w)_{(j)} b(w) = \text{Res}_z (z - w)^j [a(z), b(w)] \quad (4.18)$$

is called the j -product of $a(w)$ and $b(w)$. This product will be simply denoted by $a_{(j)} b$. By (4.16) and (4.17), the λ -bracket is related to the j -products as follows:

$$[a_\lambda b] = \sum_{j \in \mathbb{Z}^+} \frac{\lambda^j}{j!} (a_{(j)} b). \quad (4.19)$$

By defining the action of ∂ on $a(z) = \sum_{m \in \mathbb{Z}} a_m z^{-m-1} \in \mathcal{L}[[z, z^{-1}]]$ by $(\partial a)(z) = \partial_z(a(z))$, and the action of α on $a(z)$ by $\alpha(a)(z) = \sum_{m \in \mathbb{Z}} \alpha(a_m) z^{-m-1}$, we have the following results.

Proposition 4.6 *The λ -bracket satisfies the following properties:*

- (1) $[\partial a_\lambda b] = -\lambda [a_\lambda b]$, $[a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b]$.
- (2) if $(a(z), b(w))$ is a local pair, then $[a_\lambda b] = -[b_{-\lambda-\partial} a]$.

$$(3) \quad [\alpha(a)_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} \alpha(c)] + [\alpha(b)_\mu [a_\lambda c]].$$

Proof. (1) and (2) follow from [14, Proposition 2.3 (a), (b), (c)]. By (2.5) and (4.15), we have

$$[\alpha(a)(z), [b(x), c(w)]] = [\alpha(b)(x), [a(z), c(w)]] + [[a(z), b(x)], \alpha(c)(w)]. \quad (4.20)$$

This together with (4.16) gives

$$\begin{aligned} [\alpha(a)_\lambda [b_\mu c]] &= F_{z,w}^\lambda [\alpha(a)(z), F_{x,w}^\mu [b(x), c(w)]] = F_{z,w}^\lambda F_{x,w}^\mu [\alpha(a)(z), [b(x), c(w)]] \\ &= F_{z,w}^\lambda F_{x,w}^\mu ([\alpha(b)(x), [a(z), c(w)]] + [[a(z), b(x)], \alpha(c)(w)]). \end{aligned}$$

Because of $[[a(z), b(x)], \alpha(c)(w)] \in \mathcal{L}[[z, z^{-1}, x, x^{-1}, w, w^{-1}]]$, we can use the property of the formal Fourier transform, according to which $F_{z,w}^\lambda F_{x,w}^\mu = F_{x,w}^{\lambda+\mu} F_{z,x}^\lambda$. Hence,

$$\begin{aligned} [\alpha(a)_\lambda [b_\mu c]] &= F_{z,w}^\lambda F_{x,w}^\mu ([\alpha(b)(x), [a(z), c(w)]] + F_{x,w}^{\lambda+\mu} F_{z,x}^\lambda ([[a(z), b(x)], \alpha(c)(w)])) \\ &= F_{x,w}^\mu ([\alpha(b)(x), F_{z,w}^\lambda [a(z), c(w)]] + F_{x,w}^{\lambda+\mu} ([F_{z,x}^\lambda [a(z), b(x)], \alpha(c)(w)])) \\ &= [\alpha(b)_\mu [a_\lambda c]] + [[a_\lambda b]_{\lambda+\mu} \alpha(c)], \end{aligned}$$

which proves (3). □

By (4.19), Proposition 4.6 is translated in terms of j -products as follows.

Proposition 4.7 *The j -products satisfy the following equalities.*

- (1) $\partial a_{(n)} b = -n a_{(n-1)} b$, $a_{(n)} \partial b = \partial(a_{(n)} b) + n a_{(n-1)} b$.
- (2) $a_{(n)} b = -\sum_{i \geq 0} (-1)^{n+i} \frac{1}{i!} \partial^i b_{(n+i)} a$, if $(a(z), b(w))$ is a local pair.
- (3) $\alpha(a)_{(m)} b_{(n)} c = \alpha(b)_{(n)} a_{(m)} c + \sum_{i=0}^m \binom{m}{i} (a_{(i)} b)_{(m+n-i)} \alpha(c)$.

A subset $F \subset \mathcal{L}[[z, z^{-1}]]$ is called a *local family* of \mathcal{L} -valued formal distributions if all pairs of its constituents are local. The following notion of formal distribution Hom-Lie algebra can be seen as a Hom-analogue of the notion of formal distribution Lie algebra introduced by Kac [14].

Definition 4.8 Let $(\mathcal{L}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra. If there exists a local family F of \mathcal{L} -valued formal distributions, with their Fourier coefficients generating the whole \mathcal{L} , then F is said to endow \mathcal{L} with a structure of formal distribution Hom-Lie algebra. In this case, we denote \mathcal{L} by (\mathcal{L}, F) to emphasize the role of F .

Here is a simple example of formal distribution Hom-Lie algebras.

Example 4.9 Let $(\mathcal{L}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra. Denote

$$F = \{g(z) = \sum_{n \in \mathbb{Z}} g z^{-n-1} | g \in \mathcal{L}\}.$$

For any $g, h \in \mathcal{L}$, we have

$$[g(z), h(w)] = \sum_{m,n} [g, h] z^{-m-1} w^{-n-1} = [g, h] \sum_{m,k} z^{-m-1} w^{-k+m-1} = [g, h](w) \delta(z, w).$$

Thus $(g(z), h(w))$ is a local pair, and (\mathcal{L}, F) is a formal distribution Hom-Lie algebra.

The previous example shows that we can trivially associate to any Hom-Lie algebra \mathcal{L} a structure of formal distribution Hom-Lie algebra. In the following, we shall put other restrictions to the local family F , such that we can endow the formal distribution Hom-Lie algebra (\mathcal{L}, F) with a Hom-Lie conformal algebra structure.

Definition 4.10 Let $(\mathcal{L}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra. A local family $F \subset \mathcal{L}[[z, z^{-1}]]$ is called a conformal family if it is closed under their j -products and invariant under the actions of ∂ and α .

It is known from [14] that if (\mathcal{L}, F) is a formal distribution Lie (super)algebra, one can always include F in the minimal conformal family \bar{F} , such that (\mathcal{L}, F) can be given a Lie conformal algebra structure $\mathcal{R} = \mathbb{C}[\partial]\bar{F}$, with $[a_\lambda b] = F_{z,w}^\lambda[a(z), b(w)]$, $\partial = \partial_z$. For a formal distribution Hom-Lie algebra (\mathcal{L}, F) , we also have a Hom-Lie conformal algebra $\mathcal{R} = \mathbb{C}[\partial]\bar{F}$ by Propositions 4.6 and 4.7. Indeed, it suffices to extend α to $\mathcal{R} = \mathbb{C}[\partial]\bar{F}$ by $\alpha(f(\partial)u) = f(\partial)\alpha(u)$, for $f(\partial) \in \mathbb{C}[\partial]$, $u \in \bar{F}$.

The following are examples of Hom-Lie conformal algebras.

Example 4.11 Let $(\mathcal{L}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra. Denote by $\hat{\mathcal{L}} = \mathcal{L} \otimes \mathbb{C}[t, t^{-1}]$ the affization of \mathcal{L} with

$$[u \otimes t^m, v \otimes t^n] = [u, v] \otimes t^{m+n}, \quad \text{for } u, v \in L, m, n \in \mathbb{Z}.$$

Extend α to $\hat{\mathcal{L}}$ by $\alpha(u \otimes t^m) = a(u) \otimes t^m$. Then $(\hat{\mathcal{L}}, [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra. Let

$$u(z) = \sum_{n \in \mathbb{Z}} (u \otimes t^n) z^{-n-1}.$$

We have

$$\begin{aligned} [u(z), v(w)] &= \sum_{m,n} ([u, v] \otimes t^{m+n}) z^{-m-1} w^{-n-1} \\ &= \sum_{m,k} ([u, v] \otimes t^k) z^{-m-1} w^{-k+m-1} \\ &= [u, v](w) \delta(z, w), \end{aligned}$$

which is equivalent to $u_{(0)}v = [u, v] \in \mathcal{L}$, and $u_{(n)}v = 0$ for $n > 0$, namely, \mathcal{L} is closed under all the j -products. Thus, we get a Hom-Lie conformal algebra $\mathcal{R} = \mathbb{C}[\partial]\mathcal{L}$ with

$$[u_\lambda v] = [u, v], \quad \text{for } u, v \in \mathcal{L}. \quad (4.21)$$

Indeed, we can extend the λ -bracket to the whole \mathcal{R} by

$$[f(\partial)u_\lambda h(\partial)v] = f(-\lambda)h(\partial + \lambda)[u_\lambda v], \quad (4.22)$$

and extend α to a linear map of \mathcal{R} by

$$\alpha(f(\partial)u) = f(\partial)\alpha(u), \quad (4.23)$$

for any $f(\partial), h(\partial) \in \mathbb{C}[\partial]$, $u, v \in \mathcal{R}$. The conformal sesquilinearity (4.1) is naturally satisfied by (4.22), and (4.23) gives $\alpha\partial = \partial\alpha$. Thus it suffices to check (4.2) and (4.3) on the generators. Since the λ -bracket (4.21) is defined by the Hom-Lie bracket on \mathcal{L} , axioms (4.2) and (4.3) hold. So $\mathcal{R} = \mathbb{C}[\partial]\mathcal{L}$ is a Hom-Lie conformal algebra, which is viewed as a *current-like Hom-Lie conformal algebra*.

Remark 4.12 If \mathcal{R} is a free $\mathbb{C}[\partial]$ -module with the λ -bracket $[\cdot]_\lambda$ defined on a $\mathbb{C}[\partial]$ -basis of \mathcal{R} such that (4.2) and (4.3) hold, there is a unique extension of this λ -bracket via (4.1) and (4.4) to the whole \mathcal{R} (as shown in (4.22)). It is easy to show that (4.2) and (4.3) also hold for this extension. Thus, \mathcal{R} is equipped with a Hom-Lie conformal algebra structure. In the sequel, we shall often describe Hom-Lie conformal algebra structures on free $\mathbb{C}[\partial]$ -modules by giving the λ -bracket on a fixed $\mathbb{C}[\partial]$ -basis.

Example 4.13 Recall that the Virasoro conformal algebra is a free $\mathbb{C}[\partial]$ -module $Vir = \mathbb{C}[\partial]L$ generated by one symbol L such that

$$[L_\lambda L] = (\partial + 2\lambda)L. \quad (4.24)$$

Define $\alpha(L) = f(\partial)L$, with $0 \neq f(\partial) \in \mathbb{C}[\partial]$. If $(Vir, [\cdot]_\lambda, \alpha)$ forms a Hom-Lie conformal algebra, we have

$$[\alpha(L)_\lambda [L_\mu L]] = [\alpha(L)_\mu [L_\lambda L]] + [[L_\lambda L]_{\lambda+\mu} \alpha(L)],$$

which is equivalent to the following equality:

$$\begin{aligned} & f(-\lambda)(\partial + \lambda + 2\mu)(\partial + 2\lambda) \\ &= f(-\mu)(\partial + \mu + 2\lambda)(\partial + 2\mu) + (\lambda - \mu)f(\lambda + \mu + \partial)(\partial + 2\lambda + 2\mu). \end{aligned} \quad (4.25)$$

By (4.25), the highest degree of ∂ in $f(\partial)$ is at most 1. We can write $f(\partial) = a\partial + b$, with $a, b \in \mathbb{C}$ and $(a, b) \neq (0, 0)$. Comparing the coefficients of ∂^2 in (4.25), we have

$$a(\mu - \lambda)\partial^2 = a(\lambda - \mu)\partial^2, \quad (4.26)$$

which implies $a = 0$ and thus $f(\partial) = b \neq 0$. Consequently, we obtain a Hom-Lie conformal algebra $Vir = \mathbb{C}[\partial]L$ with $[L_\lambda L] = (\partial + 2\lambda)L$, and $\alpha(L) = bL$, where b is a nonzero number. We call it the *Virasoro-like Hom-Lie conformal algebra*.

With

$$[a_\lambda b] = \sum_{j \in \mathbb{Z}^+} \frac{\lambda^j}{j!} (a_{(j)} b),$$

we can give an equivalent definition of a Hom-Lie conformal algebra.

Definition 4.14 A Hom-Lie conformal algebra is a $\mathbb{C}[\partial]$ -module \mathcal{R} equipped with a linear map α such that $\alpha\partial = \partial\alpha$, and equipped with infinitely many \mathbb{C} -bilinear j -products $(a, b) \rightarrow a_{(j)}b$ with $j \in \mathbb{Z}^+$, such that the following axioms hold for $a, b, c \in \mathcal{R}$, $m, n \in \mathbb{Z}^+$:

$$a_{(n)}b = 0 \text{ for } n \text{ sufficiently large,} \quad (4.27)$$

$$\partial a_{(n)}b = -na_{(n-1)}b, \quad (4.28)$$

$$a_{(n)}b = -\sum_{i \geq 0} (-1)^{n+i} \frac{1}{i!} \partial^i b_{(n+i)}a, \quad (4.29)$$

$$\alpha(a)_{(m)}b_{(n)}c = \alpha(b)_{(n)}a_{(m)}c + \sum_{i=0}^m \binom{m}{i} (a_{(i)}b)_{(m+n-i)}\alpha(c). \quad (4.30)$$

Remark 4.15 In terms of j -products, (4.4) is equivalent to

$$a_{(n)}\partial b = \partial(a_{(n)}b) + na_{(n-1)}b,$$

which together with (4.28) shows that ∂ acts as a derivation on the j -products.

§5. Equivalence

In this section, we show that the affinization of a Hom-GD bialgebra forms a Hom-Lie algebra. Also, we prove that a Hom-GD bialgebra is equivalent to a Hom-Lie conformal algebra of degree 2.

Let \mathcal{A} be a vector space with two bilinear operations $[\cdot, \cdot]$ and \circ , and a linear endomorphism α . Denote

$$\mathcal{L}(\mathcal{A}) = \mathcal{A} \otimes \mathbb{C}[t, t^{-1}].$$

Define a bilinear operation $[-, -]$ on $\mathcal{L}(\mathcal{A})$ by

$$[u \otimes t^m, v \otimes t^n] = [u, v] \otimes t^{m+n} + mu \circ v \otimes t^{m+n-1} - nv \circ u \otimes t^{m+n-1}, \quad (5.1)$$

for all $u, v \in \mathcal{A}$, and $m, n \in \mathbb{Z}$. Moreover, define a linear map $\varphi : \mathcal{L}(\mathcal{A}) \rightarrow \mathcal{L}(\mathcal{A})$ by

$$\varphi(u \otimes t^m) = \alpha(u) \otimes t^m, \text{ for } u \in \mathcal{A}, m \in \mathbb{Z}. \quad (5.2)$$

We have the following result.

Theorem 5.1 $(\mathcal{L}(\mathcal{A}), [-, -], \varphi)$ is a Hom-Lie algebra if and only if $(\mathcal{A}, [\cdot, \cdot], \circ, \alpha)$ is a Hom-GD bialgebra.

Proof. Assume that $(\mathcal{A}, [\cdot, \cdot], \circ, \alpha)$ is a Hom-GD bialgebra. By (5.1), the bracket $[-, -]$ on $\mathcal{L}(\mathcal{A})$ is skew-symmetric since the bracket $[\cdot, \cdot]$ on \mathcal{A} is skew-symmetric. For $u, v, w \in \mathcal{A}$, $m, n, k \in \mathbb{Z}$, we have

$$\begin{aligned} & [[u \otimes t^m, v \otimes t^n], \varphi(w \otimes t^k)] + [[v \otimes t^n, w \otimes t^k], \varphi(u \otimes t^m)] + [[w \otimes t^k, u \otimes t^m], \varphi(v \otimes t^n)] \\ &= [[u \otimes t^m, v \otimes t^n], \alpha(w) \otimes t^k] + [[v \otimes t^n, w \otimes t^k], \alpha(u) \otimes t^m] + [[w \otimes t^k, u \otimes t^m], \alpha(v) \otimes t^n] \\ &= \Delta_1 \otimes t^{m+n+k} + \Delta_2 \otimes t^{m+n+k-1} + \Delta_3 \otimes t^{m+n+k-2}, \end{aligned} \quad (5.3)$$

where

$$\Delta_1 = [[u, v], \alpha(w)] + [[v, w], \alpha(u)] + [[w, u], \alpha(v)], \quad (5.4)$$

$$\begin{aligned} \Delta_2 = & m([u, v] \circ \alpha(w) + [u \circ v, \alpha(w)] - \alpha(u) \circ [v, w] - [u, w] \circ \alpha(v) - [u \circ w, \alpha(v)]) \\ & - n([v, u] \circ \alpha(w) + [v \circ u, \alpha(w)] - \alpha(v) \circ [u, w] - [v, w] \circ \alpha(u) - [v \circ w, \alpha(u)]) \\ & + k(\alpha(w) \circ [u, v] + [w, v] \circ \alpha(u) + [w \circ v, \alpha(u)] - [w, u] \circ \alpha(v) - [w \circ u, \alpha(v)]), \quad (5.5) \end{aligned}$$

$$\begin{aligned} \Delta_3 = & (m^2 - m)((u \circ v) \circ \alpha(w) - (u \circ w) \circ \alpha(v)) \\ & + (n^2 - n)((v \circ w) \circ \alpha(u) - (v \circ u) \circ \alpha(w)) \\ & + (k^2 - k)((w \circ u) \circ \alpha(v) - (w \circ v) \circ \alpha(u)) \\ & + mn((u \circ v) \circ \alpha(w) - (v \circ u) \circ \alpha(w) - \alpha(u) \circ (v \circ w) + \alpha(v) \circ (u \circ w)) \\ & + mk((w \circ u) \circ \alpha(v) - (u \circ w) \circ \alpha(v) - \alpha(w) \circ (u \circ v) + \alpha(u) \circ (w \circ v)) \\ & + nk((v \circ w) \circ \alpha(u) - (w \circ v) \circ \alpha(u) - \alpha(v) \circ (w \circ u) + \alpha(w) \circ (v \circ u)). \quad (5.6) \end{aligned}$$

By (2.5) and (5.4), we have $\Delta_1 = 0$. By (2.2), (2.3) and (5.6), we get $\Delta_3 = 0$. By (3.1) and (5.5), we have $\Delta_2 = 0$. Finally, (5.3) gives

$$[[u \otimes t^m, v \otimes t^n], \varphi(w \otimes t^k)] + [[v \otimes t^n, w \otimes t^k], \varphi(u \otimes t^m)] + [[w \otimes t^k, u \otimes t^m], \varphi(v \otimes t^n)] = 0. \quad (5.7)$$

Thus $(\mathcal{L}(\mathcal{A}), [-, -], \varphi)$ is a Hom-Lie algebra.

The arguments above are reversible. Indeed, if $(\mathcal{L}(\mathcal{A}), [-, -], \varphi)$ is a Hom-Lie algebra, then (5.7) holds, namely, (5.3) = 0. This forces $\Delta_1 = 0$ (so $(\mathcal{A}, [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra), and $\Delta_2 = \Delta_3 = 0$ for all $m, n, k \in \mathbb{Z}$. In particular, set $n = k = 0$ and $m \neq 0, 1$ in (5.5) and (5.6). In this case, $\Delta_2 = 0$ is equivalent to

$$[u, v] \circ \alpha(w) + [u \circ v, \alpha(w)] - \alpha(u) \circ [v, w] - [u, w] \circ \alpha(v) - [u \circ w, \alpha(v)] = 0,$$

and $\Delta_3 = 0$ amounts to

$$(u \circ v) \circ \alpha(w) = (u \circ w) \circ \alpha(v).$$

Set $m = n = k = 1$ in (5.6). Then $\Delta_3 = 0$ yields

$$(u \circ v) \circ \alpha(w) - (v \circ u) \circ \alpha(w) - \alpha(u) \circ (v \circ w) + \alpha(v) \circ (u \circ w) = 0.$$

Because $u, v, w \in \mathcal{A}$ are arbitrary, we conclude that $(\mathcal{A}, [\cdot, \cdot], \circ, \alpha)$ is a Hom-GD bialgebra. \square

Let $(\mathcal{A}, [\cdot, \cdot], \circ, \alpha)$ be a Hom-GD bialgebra. For $u \in \mathcal{A}$, $m \in \mathbb{Z}$, set $u[m] = u \otimes t^m$. Then (5.1) can be rewritten as

$$[u[m], v[n]] = [u, v][m + n] + m(u \circ v)[m + n - 1] - n(v \circ u)[m + n - 1], \quad (5.8)$$

for all $u, v \in \mathcal{A}$ and $m, n \in \mathbb{Z}$. Consider the following $\mathcal{L}(\mathcal{A})$ -valued formal distribution

$$u(z) = \sum_{m \in \mathbb{Z}} u[m] z^{-m-1} \in \mathcal{L}(\mathcal{A})[[z, z^{-1}]]. \quad (5.9)$$

(5.8) is equivalent to

$$\begin{aligned} [u(z_1), v(z_2)] &= [u, v](z_2)\delta(z_1 - z_2) + \partial_{z_2}(u \circ v)(z_2)\delta(z_1 - z_2) \\ &\quad + (u \circ v + v \circ u)(z_2)\partial_{z_2}\delta(z_1 - z_2). \end{aligned} \quad (5.10)$$

Furthermore, we have

$$(z_1 - z_2)^2[u(z_1), v(z_2)] = 0, \quad (5.11)$$

namely, $[u(z_1), v(z_2)]$ is local. Applying F_{z_1, z_2}^λ to (5.10), we obtain

$$F_{z_1, z_2}^\lambda([u(z_1), v(z_2)]) = [u, v](z_2) + \partial_{z_2}(u \circ v)(z_2) + \lambda(u \circ v + v \circ u)(z_2), \quad (5.12)$$

from which we can define a λ -bracket $[\cdot, \cdot]_\lambda : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}[\lambda]$ by

$$[u_\lambda v] = [u, v] + \partial(u \circ v) + \lambda(u \circ v + v \circ u), \text{ for } u, v \in \mathcal{A}, \quad (5.13)$$

where $\partial = \partial_{z_2}$. In terms j -products, (5.13) is equivalent to

$$u_{(0)}v = [u, v] + \partial(u \circ v), \quad u_{(1)}v = u \circ v + v \circ u, \quad u_{(j)}v = 0, \text{ for } u, v \in \mathcal{A}, \quad j \geq 2, \quad (5.14)$$

which shows that \mathcal{A} is closed under all the j -products. It is easy to check that the λ -bracket defined by (5.13) satisfies (4.2). However, the Hom-Jacobi identity (4.3) does not hold due to the Hom-Novikov algebra structure (\mathcal{A}, \circ) . Indeed, as we will see it, the problem can be solved by changing the order of u and v on the right-hand side of (5.13).

The following result is analogous to Gel'fand and Dorfman's statement for Gel'fand-Dorfman bialgebras and Xu's theorem for super Gel'fand-Dorfman bialgebras.

Theorem 5.2 *Let \mathcal{A} be a vector space equipped with a linear endomorphism α and two operations \circ and $[\cdot, \cdot]$. Let $\mathcal{R}_\mathcal{A} = \mathbb{C}[\partial] \otimes \mathcal{A}$ be the free $\mathbb{C}[\partial]$ -module over \mathcal{A} . Extend α to $\mathcal{R}_\mathcal{A}$ by*

$$\alpha(f(\partial) \otimes u) = f(\partial) \otimes \alpha(u), \text{ for } u \in \mathcal{A}, \quad f(\partial) \in \mathbb{C}[\partial]. \quad (5.15)$$

and define a λ -bracket $[\cdot, \cdot]_\lambda : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}[\lambda]$ by

$$[u_\lambda v] = [v, u] + \partial(v \circ u) + \lambda(v \circ u + u \circ v), \text{ for } u, v \in \mathcal{A}. \quad (5.16)$$

Then $(\mathcal{A}, [\cdot, \cdot], \circ, \alpha)$ is a Hom-GD bialgebra if and only if $(\mathcal{R}_\mathcal{A}, [\cdot, \cdot]_\lambda, \alpha)$ is a Hom-Lie conformal algebra of degree 2.

Proof. Suppose that $(\mathcal{A}, [\cdot, \cdot], \circ, \alpha)$ is a Hom-GD bialgebra. To show $(\mathcal{R}_\mathcal{A}, [\cdot, \cdot]_\lambda, \alpha)$ is a Hom-Lie conformal algebra, we need to check the axioms from Definition 4.1. The conformal sesquilinearity (4.1) is naturally satisfied as the λ -bracket is defined on generators of a free $\mathbb{C}[\partial]$ -module, as it is explained in Remark 4.12. It follows from (5.15) that $\alpha\partial = \partial\alpha$. Thus it suffices to verify (4.2) and (4.3). By (5.16) and (2.4), we have

$$\begin{aligned} [v_{-\lambda-\partial}u] &= [u, v] + \partial(u \circ v) + (-\lambda - \partial)(v \circ u + u \circ v) \\ &= [u, v] - \partial(v \circ u) - \lambda(v \circ u + u \circ v) \\ &= -[v, u] - \partial(v \circ u) - \lambda(v \circ u + u \circ v) = -[u_\lambda v], \end{aligned} \quad (5.17)$$

which proves (4.2). To show (4.3), with a direct calculation we get

$$\begin{aligned}
[\alpha(u)_\lambda[v_\mu w]] &= [[w, v], \alpha(u)] + \partial[w, v] \circ \alpha(u) + \lambda([w, v] \circ \alpha(u) + \alpha(u) \circ [w, v]) \\
&\quad + (\partial + \lambda)([w \circ v, \alpha(u)] + \partial(w \circ v) \circ \alpha(u) + \lambda((w \circ v) \circ \alpha(u) + \alpha(u) \circ (w \circ v))) \\
&\quad + \mu([v \circ w + w \circ v, \alpha(u)] + \partial(v \circ w + w \circ v) \circ \alpha(u) + \lambda(v \circ w + w \circ v) \circ \alpha(u)) \\
&\quad + \mu\lambda(\alpha(u) \circ (v \circ w + w \circ v)) \\
&= [[w, v], \alpha(u)] + \partial([w, v] \circ \alpha(u) + [w \circ v, \alpha(u)]) + \partial^2(w \circ v) \circ \alpha(u) \\
&\quad + \lambda([w, v] \circ \alpha(u) + \alpha(u) \circ [w, v] + [w \circ v, \alpha(u)]) + \mu([v \circ w + w \circ v, \alpha(u)]) \\
&\quad + \partial\lambda(2(w \circ v) \circ \alpha(u) + \alpha(u) \circ (w \circ v)) + \lambda^2((w \circ v) \circ \alpha(u) + \alpha(u) \circ (w \circ v)) \\
&\quad + \partial\mu(v \circ w + w \circ v) \circ \alpha(u) + \mu\lambda((v \circ w + w \circ v) \circ \alpha(u) + \alpha(u) \circ (v \circ w + w \circ v)),
\end{aligned}$$

where the second equality follows from (4.1) and (4.4). Similarly, we have

$$\begin{aligned}
[\alpha(v)_\mu[u_\lambda w]] &= [[w, u], \alpha(v)] + \partial([w, u] \circ \alpha(v) + [w \circ u, \alpha(v)]) + \partial^2(w \circ u) \circ \alpha(v) \\
&\quad + \mu([w, u] \circ \alpha(v) + \alpha(v) \circ [w, u] + [w \circ u, \alpha(v)]) \\
&\quad + \lambda([u \circ w + w \circ u, \alpha(v)]) + \partial\mu(2(w \circ u) \circ \alpha(v) + \alpha(v) \circ (w \circ u)) \\
&\quad + \mu^2((w \circ u) \circ \alpha(v) + \alpha(v) \circ (w \circ u)) + \partial\lambda(u \circ w + w \circ u) \circ \alpha(v) \\
&\quad + \mu\lambda((u \circ w + w \circ u) \circ \alpha(v) + \alpha(v) \circ (u \circ w + w \circ u)),
\end{aligned}$$

and

$$\begin{aligned}
[[u_\lambda v]_{\lambda+\mu}\alpha(w)] &= [\alpha(w), [v, u]] + \mu(\alpha(w) \circ [v, u] + [v, u] \circ \alpha(w) - [\alpha(w), v \circ u]) \\
&\quad + \partial\alpha(w) \circ [v, u] + \lambda(\alpha(w) \circ [v, u] + [v, u] \circ \alpha(w) + [\alpha(w), u \circ v]) \\
&\quad + \partial\lambda\alpha(w) \circ (u \circ v) + \lambda^2(\alpha(w) \circ (u \circ v) + (u \circ v) \circ \alpha(w)) \\
&\quad - \partial\mu\alpha(w) \circ (v \circ u) - \mu^2(\alpha(w) \circ (v \circ u) + (v \circ u) \circ \alpha(w)) \\
&\quad + \mu\lambda(\alpha(w) \circ (u \circ v) + (u \circ v) \circ \alpha(w) - \alpha(w) \circ (v \circ u) - (v \circ u) \circ \alpha(w)).
\end{aligned}$$

So (4.3) holds if and only if the following equalities hold

$$0 = [[w, v], \alpha(u)] - [[w, u], \alpha(v)] - [\alpha(w), [v, u]], \quad (5.18)$$

$$0 = [w, v] \circ \alpha(u) + [w \circ v, \alpha(u)] - ([w, u] \circ \alpha(v) + [w \circ u, \alpha(v)] + \alpha(w) \circ [v, u]), \quad (5.19)$$

$$0 = (w \circ v) \circ \alpha(u) - (w \circ u) \circ \alpha(v), \quad (5.20)$$

$$\begin{aligned}
0 &= [w, v] \circ \alpha(u) + \alpha(u) \circ [w, v] + [w \circ v, \alpha(u)] - [u \circ w + w \circ u, \alpha(v)] \\
&\quad - (\alpha(w) \circ [v, u] + [v, u] \circ \alpha(w) + [\alpha(w), u \circ v]),
\end{aligned} \quad (5.21)$$

$$\begin{aligned}
0 &= [v \circ w + w \circ v, \alpha(u)] - ([w, u] \circ \alpha(v) + \alpha(v) \circ [w, u] + [w \circ u, \alpha(v)]) \\
&\quad - (\alpha(w) \circ [v, u] + [v, u] \circ \alpha(w) - [\alpha(w), v \circ u]),
\end{aligned} \quad (5.22)$$

$$0 = (2(w \circ v) \circ \alpha(u) + \alpha(u) \circ (w \circ v)) - (u \circ w + w \circ u) \circ \alpha(v) - \alpha(w) \circ (u \circ v), \quad (5.23)$$

$$0 = (v \circ w + w \circ v) \circ \alpha(u) + \alpha(w) \circ (v \circ u) - (2(w \circ u) \circ \alpha(v) + \alpha(v) \circ (w \circ u)), \quad (5.24)$$

$$0 = (w \circ v) \circ \alpha(u) + \alpha(u) \circ (w \circ v) - \alpha(w) \circ (u \circ v) - (u \circ v) \circ \alpha(w), \quad (5.25)$$

$$0 = (w \circ u) \circ \alpha(v) + \alpha(v) \circ (w \circ u) - \alpha(w) \circ (v \circ u) - (v \circ u) \circ \alpha(w), \quad (5.26)$$

$$\begin{aligned} 0 &= (v \circ w + w \circ v) \circ \alpha(u) + \alpha(u) \circ (v \circ w + w \circ v) - (u \circ w + w \circ u) \circ \alpha(v) \\ &\quad - \alpha(v) \circ (u \circ w + w \circ u) - \alpha(w) \circ (u \circ v) - (u \circ v) \circ \alpha(w) \\ &\quad + \alpha(w) \circ (v \circ u) + (v \circ u) \circ \alpha(w). \end{aligned} \quad (5.27)$$

Equalities (5.18), (5.19) and (5.20) hold because of (2.5), (3.1) and (2.3), respectively. Since u, v, w are arbitrary, equalities (5.21), (5.23) and (5.25) are respectively equivalent to (5.22), (5.24) and (5.26). Furthermore, (5.21) can be rewritten as

$$\begin{aligned} &([w \circ v, \alpha(u)] + [w, v] \circ \alpha(u) - \alpha(w) \circ [v, u] - [w \circ u, \alpha(v)] - [w, u] \circ \alpha(v)) \\ &+ ([u \circ v, \alpha(w)] - [u \circ w, \alpha(v)] - [u, w] \circ \alpha(v) + [u, v] \circ \alpha(w) - \alpha(u) \circ [v, w]) = 0, \end{aligned}$$

which is implied by (3.1). By (2.3), equality (5.23) amounts to

$$(w \circ u) \circ \alpha(v) - \alpha(w) \circ (u \circ v) = (u \circ w) \alpha(v) - \alpha(u) \circ (w \circ v),$$

which holds due to (2.2). Similarly, we get (5.25). To check (5.27), we rewrite it as

$$\begin{aligned} &((v \circ w) \circ \alpha(u) - \alpha(v) \circ (w \circ u) - (w \circ v) \circ \alpha(u) + \alpha(w) \circ (v \circ u)) \\ &+ ((w \circ u) \circ \alpha(v) - \alpha(w) \circ (u \circ v) - (u \circ w) \circ \alpha(v) + \alpha(u) \circ (w \circ v)) \\ &- ((u \circ v) \circ \alpha(w) - \alpha(u) \circ (v \circ w) - (v \circ u) \circ \alpha(w) + \alpha(v) \circ (u \circ w)) = 0, \end{aligned}$$

which follows from (2.2). Thus, (4.3) holds and $(\mathcal{R}_{\mathcal{A}}, [\cdot, \cdot], \alpha)$ is a Hom-Lie conformal algebra, which is of degree 2 by (4.6) and (5.16).

Conversely, assume that $(\mathcal{R}_{\mathcal{A}}, [\cdot, \cdot], \alpha)$ is a Hom-Lie conformal algebra. The Hom-Jacobi identity (4.3) gives (5.18)–(5.27), from which we can deduce (2.2)–(2.5) and (3.1) by the discussions above. Therefore, $(\mathcal{A}, [\cdot, \cdot], \circ, \alpha)$ is a Hom-GD bialgebra. \square

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